ERIC LANDQUIST'S REFLECTIONS

Here's an idea: Suppose we get in contact with an alien civilization, but they work in a non-base ten system, then what do we do? How will we be able to communicate more easily on a mathematical level? What began as a strange and almost absurd idea started the single most rewarding academic exploration of my high school years.

For our first year at the Massachusetts Academy of Math and Science (the equivalent of a junior year at a "normal" high school), we were required to do some sort of original research project and meet with a faculty mentor to oversee the research. Josh Abrams, my mentor, had me write up a little piece on what I wanted to do and then told me to hit the library to see what other people have done in that area. My motivation—communicating with alien races—was scrapped and replaced with the more virtuous aim of exploring numbers for their sheer beauty and elegance. In any kind of research, patterns emerge, beautiful patterns, and it's only natural for a researcher to attempt to explain or prove them. This process inevitably leads to questions, further paths to head down, and it lets curiosity run wild. But to get to these paths, you must start walking. I started by walking to the library.

So there I was in front of a UNIX screen filled with text, looking at a bunch of call numbers starting with QA: the math section. The best place to start any literature search is with a historical overview of what has been done, so a book called "History of Binary and Other Non-decimal Numeration" seemed like it would work just fine. I began to skim through, looking for something, anything to catch my eye. I sifted past several patterns and research that had been performed in the past. But then I stopped and read, "Every real number can be represented canonically to any base b, where b is any real number greater than 1, and this representation is unique." Any real number greater than 1? Hmmm. My first thought was,"So e can be used as another base." ($e = 2.718281828459045...$, an irrational number.)

Within a couple hours I had my first step and several questions: "What is canonical?" "How do we use non-integer bases? They never taught us that in school!" "Well, why not base systems less than 1 or imaginary for that matter?" I continued to thumb through the literature for answers, and found work with 1.5 as a base, $\frac{1+\sqrt{5}}{2} = 1.618033...$ (also known as the golden ratio), and $-1 + i$. They were fun to play with, since it was stuff I'd never seen before, and yet they were elementary enough for a high school student to understand. Some journal articles took some time, and a lot of help from Josh, to finally tackle.

My own notes were stained with macaroni and cheese as I played with numbers during dinner or the weekends. The mathematics was starting to get addictive. I started off with rational base systems: 2, 2.1, 2.2, 2.3, 2.4, etc., up to 3, throwing in e just for fun. I figured that expressing integers (like 3) in these bases would give me some nice patterns. On a Boy Scout biking trip to Cape Cod, I spent the entire car ride up and back, and considerable time in my tent before bed with my TI-85 in hand, crunching out $3 = 10.111111001001111010001...$ in base 2.1, and in the other bases until I would find something. But the surprise was that in all these rational bases, an integer appeared to be irrational: an endless non-repeating set of digits.

Here's where the breakthrough came. I asked myself "In all these bases, except 2 and 3, 3 appears to be irrational, so in what bases is 3 obviously rational?" Let's find such a base! Let $3 = 10.1$ to some base b. Setting up the equation, $3 = b^2 + \frac{1}{b}$, I solved to find that $b = \frac{3+\sqrt{5}}{2} = 2.6180339887....$ Hey, that's irrational! Similar tests showed that other such equations produced solutions that were irrational bases. I had to tell Josh! Josh was as surprised as I was with the results I had thus far. We fed on each other's curiosity and genuine interest in the research, and came up with more questions to further the results. I started looking into fractions expressed in these irrational bases, and saw that they had repeating expansions, like $\frac{1}{7} = 0.04453453453...$ in base $3 + \sqrt{8}$. For a week or so, I just crunched numbers, amazed at the patterns. But I knew that all these irrational bases had to form some pattern themselves. Was there a formula that would produce these for me?

The setting is the annual church Turkey Bowl touch football game. I started throwing the football around with my brother as numbers went flying around my head. I stopped all of a sudden, ran into the youth center and up to the white board, and began to feverishly write up a bunch of formulas that I had in my head. My brother was mumbling "We're playing football, and you start doing all this stupid math. DAD! Eric's doing math again!" I knew I had the formula I was looking for: $b = \frac{a\pm\sqrt{a^2\pm4}}{2}$, where a is an integer. These bases produce finite decimal expansions for all integers! I tried some examples, and found that the formula produced the same base systems I was getting before. Cool, and shockingly similar to the quadratic formula for solving second-degree polynomials! I had a fun time trying to explain all the numbers on the white board in Sunday School the next morning!

Again, I went to Josh on Monday morning and proudly displayed the formula. For the next several months, I found pattern after pattern: palindromes; a denominator of 2 whenever you raise my formula to a positive integer power; and big triangles with Pascal's Triangle mysteriously imbedded. Yet I was haunted with the still open question of why all these irrational bases kept giving me finite representations for integers and repeating representations for rational numbers. I thought that the fact that powers of that formula yielded a denominator of 2 had something to do with it, that all the irrational terms would have to then cancel out, somehow. I had difficulty proving this, and attempted many times, but with no success. Josh had an idea, but rather than tell me the answer, he taught me some notation that I'd need and some math I didn't know yet.

We took off to the library and sat down with the formula and a table of numbers I calculated based on powers of my formula. Bit by bit, Josh guided me through deriving formulas for each row and column in the table. He was able to find just the right balance so that he could get the most out of my research without intruding on my work, letting me call this monster formula that described the entire table mine. I could immediately put that monster formula into my proof and it clicked. In those minutes in the library, I had a complete proof by induction that the formula $\frac{a\pm\sqrt{a^2\pm4}}{2}$, when raised to any integer power, has a denominator of 2. Josh and I came back from the library, and in huge letters, Josh wrote, "Eric has his lemma proven!" on the whiteboard of the "Fishbowl," the main room at the Academy. It was a proud moment.

The regional science fair was now coming up and I was one of 12 picked from the school to enter. To my surprise, I won the fair. Because of that performance, I was invited to a judge-off in Boston with the five other winners of the regional fairs in Massachusetts, and then to compete as a delegate to the International Science and Engineering Fair being held in Hamilton, Ontario, Canada. Over a thousand of the best science and engineering projects from all over the world covered the coliseum at the International Fair and this crazy idea about irrational base systems was one of them! To my delight there were also a whole lot of other math projects there! This gave me a good opportunity to chat with others about their projects, including a student from Denmark. We kept in touch for a while after that, talking about math, and sharing research with each other. At the fair, I was also able to get the name of a professor at the University of Waterloo who worked with complex base systems, including using these bases to create fractals. So through the research I did, I was able to make a lot of contacts and get a bigger idea of research that was being done.

At this point, I left behind the thought of trying to prove my biggest conjecture and focused instead on studying my material to be ready for judging. I knew I would be talking with university professors, mathematicians from industry, and representatives of the American Mathematical Society. As I looked back at some of my first few notes, a light went on. I began to read about an addition algorithm in base $\frac{1+\sqrt{5}}{2}$ by George Bergman. Then it hit me! I could extend that algorithm to include any bases of the form $\frac{a\pm\sqrt{a^2\pm 4}}{2}$ to prove that all base ten integers have finite representations in these bases. Since there were finitely many steps in this algorithm, and none of the steps put an infinite number of digits on the representation, then I proved it. YES! Ah the wonders of stuckness and the beauty of proof. Since this description was fairly rough, I could only present a basic outline for how the algorithm worked and that I essentially had the proof. I was still a little shaky as to why rational numbers have repeating expansions, but a division algorithm would later prove that. For the icing on the cake, I earned a fourth-place award.

For senior year, we were not required to do a research project, but I wanted to go back to the International Fair. I wanted to explore some other lingering questions. There were several questions I hadn't answered and had actually left in the dust. Josh and I had talked about every question I came across. We came up with some questions to pursue in order to help with the focus and flavor of the problem. So immediately after the International Fair, I set to work on designing a division algorithm and playing with bases I hadn't before. If irrational bases are bizarre, then negative bases (bases whose absolute value is less than 1) and complex bases involving irrational numbers are simply inhuman. Work in these bases is completely impractical for any kind of application, and if it were practical, then I probably wouldn't care. But the joy of working with the craziest of numbers is just that: it's theoretical, beautiful, and is something few people consider. Plus the patterns that kept emerging were really cool.

Another joy about continuing a project like this is that you know what has worked in previous years, so you are better able to find and prove interesting results faster. My biggest joy that senior year was the discovery of a set of positive irrational bases that produced similar results as before: integers with finite representations and rationals with repeating representations in these bases. I quickly found and proved several patterns with this family, which is the set of all numbers

$m + \sqrt{n}$ where $(m-1)^2 < n < (m+1)^2$. This formed the bulk of my senior year research.

After high school, I went to Virginia Tech (Go Hokies!) as a Math major and have since earned a Master's. With all the research I did as a graduate student and undergraduate, I learned the most from working with irrational bases in high school. It gave me a love for math (number theory in particular) and a desire to do research. If there is any way to have an emotional connection to math, then my relationship with irrational bases is a prime example. Next fall I plan to enroll in a Ph.D. program in number theory. I hope to get the chance to extend my work with irrational bases to a dissertation level. If I do, I should put a big alien head on the front page. QED.

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