DAN MCGINN'S REFLECTIONS

My eighth grade algebra teacher once gave us a problem to think about called The Emperors Oats. In order to choose which suitor would marry his daughter, the Emperor seats n suitors around a table, and proceeds to dump oats on the head of every other suitor. If he comes across someone who already has oats on his head, the Emperor does not count him. He continues in this way until only one lucky suitor is left "oatless" and the Emperor awards that suitor the hand of his daughter.

The question is: if you know ahead of time where the Emperor will start his counting, can you find a rule that, if you know the number of suitors, you can determine where the groom-to-be is seated? Eventually I came up with an answer that I could explain, but I couldnt express it as an equation. The next day, the teacher showed us how cool things happen if you look at the answers in binary.

Although not terribly complex, this was the first math problem I worked on where the answer wasn't "7 apples" or "$x = \frac{1}{3}$," so this problem sort of stuck out in my mind. So, when it came time to choose a topic for a mathematics research project, I thought that I might look into this problem a little more.

As I researched The Emperors Oats problem, it seemed as if it was just about solved. Then I found out that it was just a simple case of a more general problem—The Josephus Problem (named after a famous Bible historian). The Josephus Problem actually has a pretty bloody history. In this version, there is a circle of n people, and every kth person is "removed." (The Emperors Oats was the specific case with $k = 2$). I was curious to see if the binary trick my teacher had suggested worked for other values of k.

One of the first steps I took was to see what other people had found out about this problem. Reading mathematics papers can be an arduous task. It turns out that there is actually very little written on this topic. Specifically, my research showed no mention of patterns in other bases. In fact, it seemed like there was very little known about these kinds of patterns in general, with the exception of one extensive and complicated equation that could approximate answers fairly closely (depending on how many decimal places of some abstract constant you used). After making tables, writing computer programs, and drawing countless circles of numbers, it became clear that (besides the special case of $k = 2$) the patterns formed in other bases were just as complex as the patterns in base 10.

Having worked on this problem for so long, I had become pretty familiar with it. There seemed to be some sort of underlying organization to the pattern but I had no idea what was causing it or how to isolate it. So, I began to investigate other facets of the problem. I began looking at not just the last person left "oatless" but at the second to last as well. Later, I looked at the orders in which all "suitors" are removed. So, for example, the order of removal for $n = 5$ (meaning there are 5 suitors) and $k = 3$ (meaning you remove every third one) is denoted $J(5, 3)$. For $J(5, 3)$, the removal pattern is (3, 1, 5, 2, 4), which means that the third suitor is removed first, the first suitor in the circle is removed next, the fifth is removed next, and so forth. And for $J(6, 7)$, the pattern is (1, 3, 6, 2, 4, 5).

I then asked:

- For a given n and k, is there a pattern between the second to last person to be removed and the last person?
- What happens if you keep n steady and vary k?
- What happens if you keep k steady and vary n?

Eventually it occurred to me that for a given n there are only so many orders in which to remove n people (at most $n!$). And sure enough, I discovered that there is a cyclic pattern here. If you have n people and remove every kth person, everyone gets removed in the same order as when there are n people and you remove every $(k + n!)$th person. That is, $J(n, k) = J(n, k + n!)$. In a sense, after you have gone through $n!$ different ks, you have run out of the possible orders of removal and have to start over again.

I'm still working on this problem. My conjectures have explanatory proofs, but not rigorous ones. It turns out that the repeating period is often less than $n!$. My conjecture is that $J(n, k) = J(n, k+l\text{cm}(2, 3, 4, \ldots, n))$. As I've gotten farther along in my mathematics work, it seems that the patterns I've found are closely related to abstract algebra. For example, the set of all possible orders of removal for 4 people is actually the alternating group of order 4. The same is true for 5 people, but the set of all possible orders of removal for 6 people does not form a group. Currently, Im focusing on trying to figure out for what other n, if any, does the set of all possible orders of removal form a group.

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