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Tackling twisted hoops

Untangle these wire pretzels

by S. Matveyev

LET'S MAKE A CIRCLE OUT OF THIN WIRE, smoothly curve it to give it a more complex shape, and flatten it against a plane (figure 1). What we have is a flat, tangled hoop. *Is it possible to disentangle the wire hoop to obtain the circle again without lifting it from the plane?*

We assume that the wire has zero thickness, so that at the points where one section of the wire passes above another (we'll call them *double points*), the upper section also lies on the plane. The wire is very flexible, but not in-

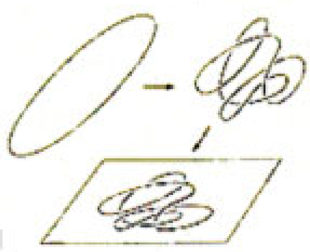
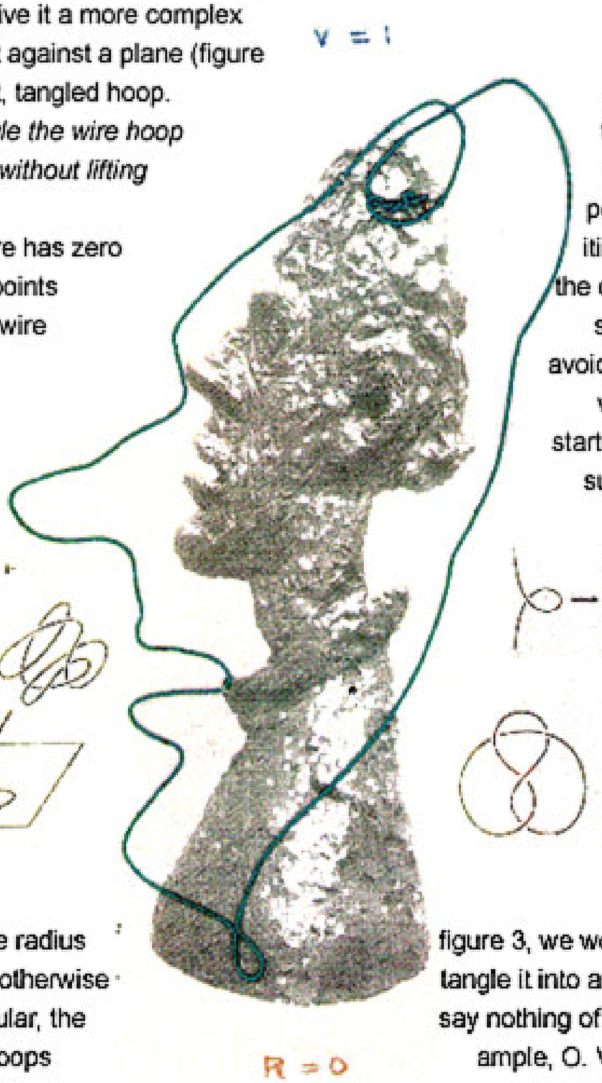
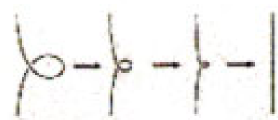


Figure 1



finitely flexible, so that the radius of curvature is not zero - otherwise the wire breaks. In particular, the method of straightening loops shown in figure 2 is prohibited.

We know that it's possible to disentangle the hoop in space. After all, we obtained the tangled hoop from a wire circle. If we perform the same operaitions in reverse, we'll get the original circle. Since we started from a circle, we avoided all the difficulties involving knots - if we had started from a knotted hoop, such as the one shown in



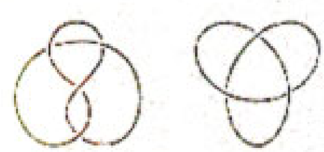


figure 3, we would be unable to disentangle it into a circle even in space, to say nothing of the plane. (See, for example, O. Viro's article in the May/June 1998 issue of Quantum).

Experiment a bit, think a bit

Let's begin with the wire hoop shown in figure 1. If you play around a bit with a piece of wire (or thread), you'll see that this hoop can be disentangled (see figure 4).

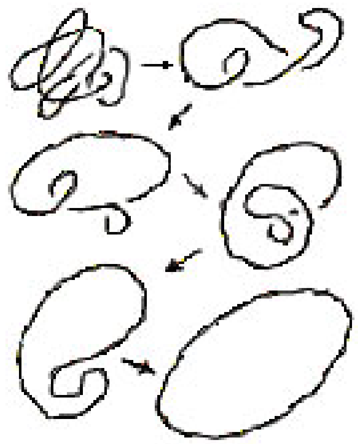


Figure 4

Now I invite the reader to disentangle hoops a-f in figure 5.

I hope you succeeded with hoops b, e, and f. But your inability to disentangle hoops a, c, and d should convince you that hoops exist that cannot be disentangled. How can we prove that a particular hoop cannot be disentangled?

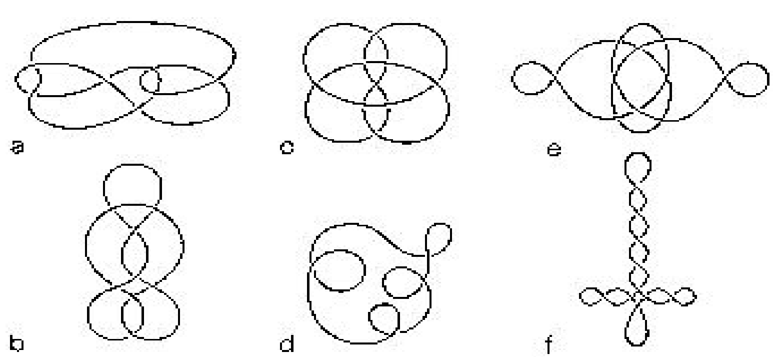
To prove that a certain construction or process is impossible, mathematicians often use the following remarkable method. Every state of the object under consideration is assigned a number that remains the same throughout the process (such a number is called an *invariant*). Then the invariant is determined for the initial and the desired states of the object. If different values are obtained, it means that it is not possible to pass from the initial state to the desired state-after all, that's why it's called an invariant: it cannot change during the process!

So let's try to assign a number to every hoop on the plane. The first idea that comes to mind is to count the double points in the hoop. Alas! this is not an invariant, as you can see from figure 4. However, examining this figure, we notice that double points appear and disappear in pairs. This leads to the idea that the parity of the number of double points is an invariant (in other words, the remainder upon division of the number of double points by two is an invariant).

This is indeed the case, as we will see later. This fact implies, for example, that the hoop in figure 5a cannot be disentangled (it has seven double points, while the circle has no double points-this prevents us from transforming the hoop into the circle). The hoop in figure 5d has four double points; thus its invariant is zero, the same as for the circle. Does this mean that it can be disentangled? No, it does not, because we don't know whether the condition of zero invariants is sufficient. Thus the question of whether the hoop in figure 5d can be disentangled remains open.

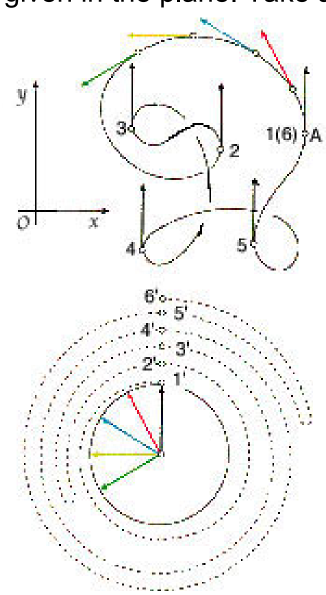
The reasoning above must convince you that it makes sense to search for invariants in this case. Let's do just that.

Figure 5



The invariant *V*

Let a tangled hoop be given in the plane. Take an



arbitrary point A on this hoop and choose one of two possible directions of going around the hoop. We'll move a point along the hoop, starting at point A, with unit speed in this direction. The velocity vector will turn about A' and its endpoint will move along a circle centered at A'. When we complete the tour around the hoop and return to point A, the velocity vector returns to its initial state; therefore, the total number of revolutions of this vector about A' is an integer. We assign revolutions made in the positive direction (counterclockwise) a plus sign and revolutions made in the negative direction (clockwise) a minus sign.

Look at figure 6. In this figure, the endpoint of the velocity vector is shown as a dashed curve and is shifted from the circle to make it easier to see what's going on. In reality, the red curve is tightly wound on the circle, and points 1' – 6' coincide. All in all, the velocity vector performs -1 revolution: from point 1' to point 2', one revolution; from point 2' to point 3' and from 5' to 6', no revolutions; from point 3' to point 4', one revolution in the negative direction; and from 4' to 5', one revolution in the negative direction as well.

The invariant we promised (we'll denote it by *V*) equals the absolute value of the total number of revolutions of the velocity vector. It's clearly independent of the choice of the starting point *A*, nor does it depend on the initial direction taken; indeed, changing the direction merely changes the sign of the total number of revolutions. For example, for the hoop in figure 6, the invariant is 1.

We'll show (without giving a rigorous proof) that *V* is actually an invariant. When the hoop is disentangled, the position of the velocity vector changes smoothly, without making any jumps. Therefore, the number *V* must also change smoothly. However, *V* is an integer and it can turn into another integer only by making a jump, which contradicts the criterion of continuity. Therefore, *V* remains unchanged and is indeed an invariant of the disentangling operation.

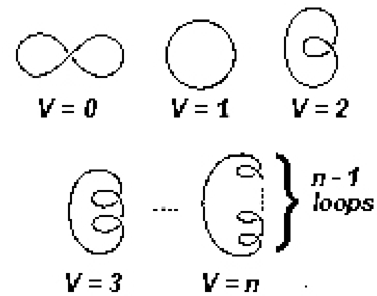
Now we can tackle the hoop in figure 5d. For this hoop, *V* = 3 (check this on your own!); therefore, it cannot be disentangled into a circle, for which V = 1.

If you actually verified that *V* = 3 for the hoop in figure 5d, you must have noticed that, in practice, it's not so easy to calculate the number of revolutions of the velocity vector. In fact, it's easy to miscalculate. However, there's an easier way to calculate *V*.

For this purpose we choose a direction in the plane-for example, the direction of the axis *Oy* (see figure 6) - and mark the points of the hoop where the velocity is parallel to *Oy* and in the same direction. We write the number +1 near a marked point if the small section of the hoop containing this point lies to the left of it; we write the number -1 near a marked point if the section of the hoop containing this point lies to the right of it. (If the section containing the marked point lies on *both* sides of it, we don't write any number. This happens when the vector is traveling along a loop and suddenly starts looping in the other direction at the marked point - it traces a sort of flattened *"S."* and never completes the first loop.) Now we can say *the invariant V equals the absolute value of the sum of all the numbers written.*

For example, in figure 6, we write + 1 at points 1 and 2 and -1 at points 3, 4, and 5. Thus *V* = 1 for this hoop.

Figure 7



We invite the reader to prove that this method actually gives the value of V for any hoop.

In figure 7, for any nonnegative integer *n,* a hoop whose invariant *V* equals *n* is shown. We recall that if a hoop can be disentangled into a circle, its invariant *V* must be equal to the invariant of the circle — that is, to 1.

The invariant *R*

The equality *V* = 1 is a necessary condition for a hoop to be disentangled into a circle. But is this condition also sufficient? At first I thought it was, but unsuccessful attempts to disentangle my belt, arranged as shown in figure 8b, convinced me that it wasn't and simultaneously elicited an important observation: when I picked the belt up off the floor (figure 8a), it was twisted completely around twice!

Let's replace the hoop with a band that lies on the plane such that its middle line coincides with the hoop

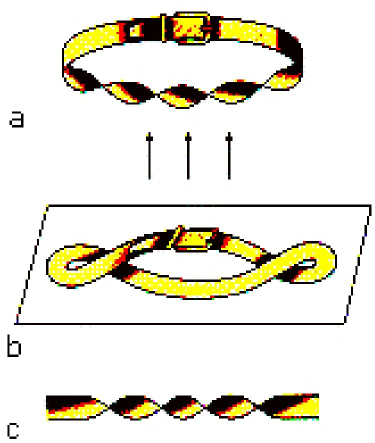


Figure 8

(figure 9a). Disentangling the hoop in space (for example, returning it to the initial state in which it was before placing it on the plane), we obtain a twisted band. We denote the number of complete twists (where the "front" is twisted around and faces the front again) by R. This number is our *second*

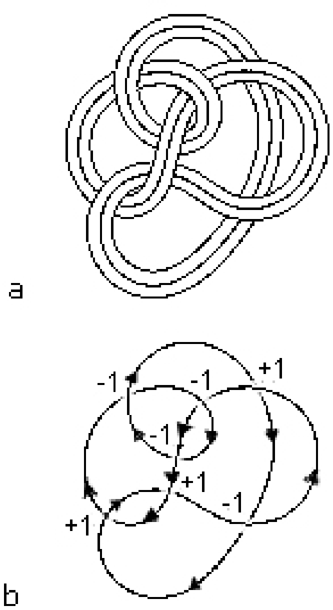


Figure 9

invariant, and if the hoop is to be disentangled into a circle, this invariant must be zero.

To be more specific, the number of complete twists is given a plus sign if the band is twisted as in figure 8a, and a minus sign if it's twisted as in figure 8c (recall the difference between a left- and right-threaded screw).

We'll prove that *the number R is indeed an invariant* – that is, it doesn't change when the hoop is disentangled in the plane. It's sufficient to notice that disentangling the hoop determines a method for disentangling the corresponding band. But the number of twists of the band remains unchanged not only when the band is disentangled in the plane, but for any three-dimensional motion.

It can be proved (we won't do it here) that the invariant *R* can be calculated as follows. Choose a direction for going around the hoop. Then mark every double point with the number +1 if the lower velocity vector is directed to the left of the upper velocity vector; otherwise, mark this double point with -1. It's easy to see that these numbers are independent of the direction chosen. The invariant *R* equals the sum of these numbers. For example, the hoop in figure 9b has three positive and four negative double points; thus, its invariant *R* is -1. Therefore, this hoop cannot be disentangled in the plane.

Necessary and sufficient conditions

We've already seen that the conditions $V = 1$ and $R = 0$ are necessary for the hoop to be disentangled into a circle. But are these conditions sufficient? In other words, is it sufficient to check that $V = 1$ and $R = 0$ to be sure that the hoop can be disentangled into a circle? The answer is yes.

Fundamental theorem. *In order for a hoop to be disentangled into a circle in the plane, it is necessary and sufficient that its invariant V be equal to* 1 *and its invariant R be zero.*

This theorem gives the complete answer to the question formulated at the beginning of this article. The simple methods described above for evaluating *V* and *R* allow us to quickly check the necessary and sufficient conditions in the theorem. We invite the reader to apply this theorem to the hoops depicted in figure 10.

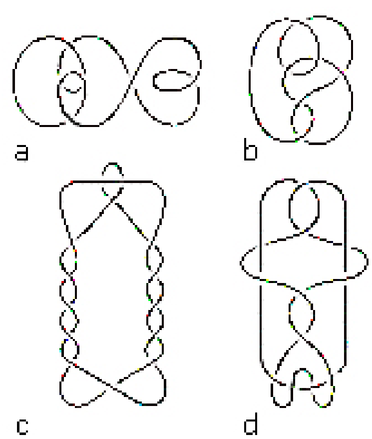


Figure 10

Proof of the fundamental theorem

We've already proved that *V* and *R* are invariants; thus the necessity of the conditions $V = 1$ and $R = 0$ is already proved. To prove sufficiency, we must showthat every hoop for which $V = 1$ and $R = 0$ can be disentangled into a circle in the plane.

Consider a hoop of this type. We know that it can be disentangled in three-dimensional space. We denote by $\tilde{K}_t$ the position of the hoop at the time *t* in the process of disentangling it. The moment *t will* be called *singular* if the hoop $\tilde{K}_t$ has a vertical tangent at one or more of its points. Assume that there are no singular moments. Then the hoop can be disentangled in the plane. Indeed, assume the ceiling of the room where we work with the hoop is parallel to the plane to which the hoop belongs. Imagine that the ceiling starts dropping until it reaches the plane with the hoop. In the process, every hoop $\tilde{K}_t$ goes to a certain plane hoop K_t. The absence of vertical tangents guarantees that no folds (points with zero curvature) occur in the hoops K_t. The family of hoops K_t determines the desired method for disentangling the given hoop into a circle in the plane.

Now consider how the hoop K_t behaves when the moment $t = t_0$ *is* singular — that is, the hoop passes the state K_{t_0} with a vertical tangent. A typical picture of this passage through the vertical state is shown in figure 11. We see that when the hoop undergoes the transformation $\tilde{K}_{t_1} \to \tilde{K}_{t_0} \to \tilde{K}_{t_2}$ in space, the corresponding plane hoop undergoes the forbidden transformation $\tilde{K}_{t_1} \to \tilde{K}_{t_0} \to \tilde{K}_{t_2}$ during which the break K_{t_0} occurs and a loop appears on the hoop K_{t_2}.

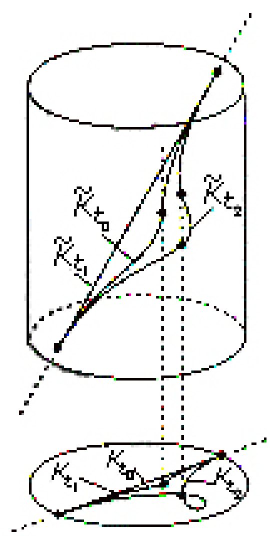


Figure 11

It can be proved (but not here) that the process of disentangling any hoop in space can be performed in such a way that only a finite number of singular moments occurs and all of them are typical — that is, *a single loop appears or disappears at each of these moments*.

Now assume that a loop has *appeared* at a singular moment. We cannot create a loop by transforming the hoop in the plane, but we can create two (mutually annihilating) loops, as shown in figure 12. Thus, we create two loops, contract the extra one into a very small loop, and "freeze" it.

Continuing the process of disentangling simultaneously in three-dimensional space and in the projection onto the plane, we transform the plane hoop

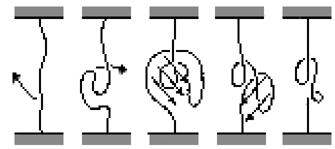


Figure 12

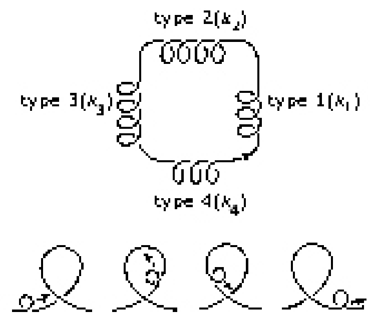


Figure 13

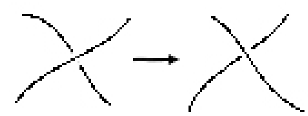


Figure 14

into a circle with a finite number of small ("frozen") loops. These loops can be classified into four types depending on where the loop is situated (inside the circle or outside of it) and in what order its double point passes (first the upper and then the lower thread, or vice versa). Then we can change the order of the loops by pulling one through the other, as shown in figure 13.

If k_i denotes the number of loops of type i, then $V = 1 + k_1 + k_2 - k_3 - k_4$ and $R = k_1 - k_2 + k_3 - k_4$. Recalling that $V = 1$ and $R = 0$, we obtain the system of equations

$$\begin{cases} k_1 + k_2 - k_3 - k_4 = 0 \\ k_1 - k_2 + k_3 - k_4 = 0 \end{cases}$$

from which it follows that $k_1 = k_4$ and $k_2 = k_3$. A pair of loops of types 1 and 4 can easily be destroyed, as shown in figure 12; the same is true for pairs of loops of types 2 and 3 (figure 4). It remains to transform our circle with loops into a real circle. The theorem is thus proved.

Disentangling hoops with self-intersections

Now let's change the statement of the problem by saying we're allowed to create self –intersections while we're disentangling the hoop. More precisely, we're allowed to pull the lower part of the loop through the upper part near double points, as shown in figure 14. This problem statement doesn't seem quite natural (indeed, to perform such a transformation, we must cut the hoop and glue it back together, which can wear down even the most patient experimenter). And yet a formal mathematical problem investigated by the American mathematician H. Whitney in the 1930s can be reduced to this very statement. In fact, Whitney's problem served as the starting point for this article.

Since we are now interested in disentangling hoops in the plane with self-intersections allowed, the reader is invited to prove the following statements.

1. The number k_i is an invariant of the operation of disentangling with self -intersections. (Hint: recall the method for evaluating *V* described above.)

2. The number R is not an invariant of the operation of disentangling with self -intersections. (Hint: experiment with a belt, exchanging the upper and lower parts near one of the double points.)

3. The remainder *R'* upon division *of R* by 2 is an invariant of the operation of disentangling with self -intersections. (Hint: every operation of self -intersection replaces the number ±1 marking the double point with $\mp$1.)

4. The number *R'* is an invariant of the operation of disentangling (without self -intersections!) the hoop in the plane.

To make further progress, we'll need the notion of a *simple loop*: this is a portion of the hoop that begins at a double point, ends at the same double point, and has no self-intersections (though it may intersect other portions of the hoop, as shown in figure 15). Now try to prove the following series of propositions.

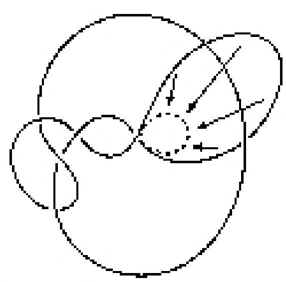


Figure 15

5. Every plane hoop has a simple loop.

6. Every simple loop can be contracted (with self -intersections!) into a small loop without affecting other parts of the hoop.

7. Any hoop can be transformed (with self -intersections) into a figure eight, a circle, or a circle with a finite number of small loops inside it.

8. Any hoop can be transformed (with self -intersections) into any other hoop if we first add to one of these hoops several (how many?) loops.

9. (Whitney's theorem) A hoop with invariant V_1 can be transformed into another hoop with invariant V_2 if and only if $V_1 = V_2$.

In conclusion, we present three more problems related to the initial problem statement (concerning the process of disentangling without self -intersections).

10. For any pair of integers *m* and *n* with an odd sum ($m \geq 0$), construct a hoop with invariants *V* = *m* and *R* = *n*. Why don't any hoops exist with invariants V = 1 and R = 1?

11. Formulate and prove an analogue of Whitney's theorem for disentangling hoops without self -intersections.

12. Prove that any hoop on the sphere can be transformed (without self-intersections) into either a circle or a figure eight.